Radiation by charged particles in nonuniform acceleration: The inapplicability of Larmor's formula

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A recent work which explained synchrotron radiation as an inverse Compton effect [R. Lieu and W. I. Axford, Astrophys. J. **416**, 700 (1993)] highlighted the importance of coherence in characterizing the radiation from a relativistic electron. In this paper we demonstrate explicitly that the well-known Larmor formula for the classical radiative loss rate of a circulating charge is only valid in the limit of constant acceleration. When the acceleration varies significantly over a ''radiation formation length,'' the radiation properties are determined by coherent interference in an orbit-dependent way, and spectra and total loss rates are modified nontrivially. We illustrate these effects by considering a representative trajectory where analytical results are available. [S1063-651X(96)01812-0]

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The relativistic generalization of Larmor's formula [2] for the radiative energy loss rate of an electron accelerated transversely at velocity $\mathbf{v}$ is given by

$$\frac{dE}{dt}=\frac{2}{3}\frac{e^2}{c}\gamma^4\left(\frac{d\mathbf{v}}{dt}\right)^2, \tag{1}$$

where $\gamma=(1-v^2/c^2)^{-1/2}$. This result has widely been used to calculate losses in astrophysical plasmas and in circular laboratory accelerators. It can be integrated with respect to time to produce an average loss rate along the electron trajectory in the limit of constant or slowly varying acceleration. Thus, e.g., the brightness of nonthermal emission from supernova remnants is modeled by a power-law energy distribution of relativistic electrons and a mean magnetic field [3]. Although it is well known that such fields are far from smooth, more realistic treatments invariably assume that it remains smooth in small spatial scales [4].

In this paper we demonstrate that Eq. (1), though correct instantaneously, can lead to large errors if it is applied to calculate radiative loss rates in situations when the acceleration varies over length scales smaller than the so-called ''radiation formation length'' $2r/\gamma$, where r is the orbital radius of curvature at any point. Owing to the coherent addition of the genuinely *varying* radiation amplitudes from different parts of the formation length, the resulting loss rate is not necessarily given by the instantaneous Poynting flux at any time. Such small scale field variations are commonplace, relevant phenomena. In laboratory accelerators, such as the Hadron Electron Ring Accelerator (HERA) bending magnets at DESY, $2r/\gamma$ is about a few cm, but wigglers or undulators with fields varying over wavelengths shorter than 1 cm can be present in sections of any synchrotron beamline. Consider another example, viz., synchrotron radiation by cosmic ray electrons in the magnetic field B of our galaxy. The distance $2r/\gamma$ given above then equals $2m_ec^2/eB$ (motion along the field is ignored). It is generally believed that the interstellar field is frozen into a thermal plasma of temperature $T<10^4$ K, and can therefore be turbulent over distances as small as $(2m_pkT)^{1/2}c/eB$, the gyroradius of a proton. The ratio of the former distance to the latter is >10; again the field is not necessarily smooth from the radiation viewpoint.

We will calculate the radiative loss rate for a two-dimensional synchrotron orbit, with spatial coherence effects explicitly taken into account. This approach will give correct results for field inhomogeneities at all length scales, which can then be compared with (1). As starting point we use the classical formula for the emissivity, defined here as the energy output per unit solid angle per unit angular frequency, when a beam of radiation intercepts the observer's line of sight [2]:

$$\frac{d^2E}{d\omega d\Omega}=\frac{e^2\omega^2}{4\pi^2c^3}\left[\left|\int v_y e^{i\omega(\tau-x\cos\theta/c)}d\tau\right|^2+\sin^2\theta\left|\int v_x e^{i\omega(\tau-x\cos\theta/c)}d\tau\right|^2\right]. \tag{2}$$

The integral over time τ can be interpreted as a coherent summation of outgoing radiation amplitudes from all parts of the particle's orbit [1,5].

We employ a powerful mathematical tool that gives good approximations to the integrals in (2) for the most important range of synchrotron radiation frequencies. The method evaluates a generic form:

$$F(\omega)=\int_{-\infty}^{\infty}f(\tau)e^{i\omega\psi(\tau)}d\tau \tag{3}$$

in the case of large ω and $\dot{\psi}(\tau)>0$ for all real values of τ, by choosing a path of integration in the complex plane which maintains a small but positive imaginary part. It can then be shown [6] that $F(\omega)$ is vanishingly small except when the phase is stationary at $\dot{\psi}(\tau_s)=0$, around which a path of steepest descent must be constructed as a ''link'' to result in finite contributions. The standard version of this method [6] treats the simple case when (a) τ_s is unique and (b) the real part of $\ddot{\psi}$ at $\tau=\tau_s$, viz., $\ddot{\psi}_r(\tau_s)$, vanishes. It gives

$$F(\omega)\simeq\left[\frac{2\pi}{\omega\ddot{\psi}_i(\tau_s)}\right]^{1/2} f(\tau_s)e^{-\omega\psi_i(\tau_s)}, \tag{4}$$

where ψ_i denotes the imaginary part of ψ.

Equation (4) is appropriate to solving the two-dimensional constant acceleration problem since criteria (a) and (b) are both satisfied. We can write

$$v_x=v\cos\Omega_o\tau, \quad v_y=v\sin\Omega_o\tau, \quad x=y=0 \text{ at } \tau=0 \tag{5}$$

and locate the observer by the elevation and azimuthal angles (θ,ϕ), both measured with respect to the x axis. As is well known, we can then expand the electron position and velocity in powers of τ, with the time origin t given by

$$\phi=\Omega_0 t. \tag{6}$$

For the radiative loss rate at a specific instance of time, we may define the observational azimuth accordingly such that $t=0$. This simplifies the algebra by ignoring parts of the orbit which are not coherently related to the point of interest.

To $O(1/\gamma^2)$ Eq. (2) now reads

$$\frac{d^2E}{d\omega d\Omega}=\frac{e^2\omega^2}{4\pi^2 c}\left[\left|\int_{-\infty}^{\infty}\Omega_0\tau e^{i\omega\psi(\tau)}d\tau\right|^2 + \theta^2\left|\int_{-\infty}^{\infty}e^{i\omega\psi(\tau)}d\tau\right|^2\right],$$

where

$$\psi(\tau)=\frac{1}{2}\left[\left(\frac{1}{\gamma^2}+\theta^2\right)\tau+\frac{\Omega_0\tau^3}{3}\right].$$

Now $\dot{\psi}(\tau_s)=0$ when

$$\tau_s=\frac{i}{\Omega_0}\left(\frac{1}{\gamma^2}+\theta^2\right)^{1/2}.$$

Substitution in (4) then leads to an approximate value for the emissivity given by

$$\frac{d^2E}{d\omega d\Omega}\simeq\frac{e^2\omega}{2\pi c\Omega_0}\left(\frac{1}{\gamma^2}+\theta^2\right)^{1/2}\left(1+\frac{\theta^2}{1/\gamma^2+\theta^2}\right) \times e^{-(2\omega/3\omega_0)(1/\gamma^2+\theta^2)^{3/2}}.$$

With the help of tables [7], integrations can be performed over ω and the θ part of the solid angle, viz., $\int_0^\infty d\omega\int_{-\pi/2}^{\pi/2}\cos\theta d\theta$. The result, $dE/d\phi$, is converted to a total radiative loss rate dE/dt instantaneous at the time origin $t=0$ by using the relation $(d\phi/dt)=\Omega_0$, which is a consequence of (6). We obtain

$$\left(\frac{dE}{dt}\right)_{t=0}=\Omega_0\frac{dE}{d\phi}\simeq\frac{9}{5\pi}\frac{e^2}{c}\gamma^4\Omega_0^2.$$

The approximate formula derived above can be compared with Larmor's theorem, where the corresponding loss rate obtained by applying (1) to the same situation (i.e., by setting $|d\mathbf{v}/dt|=v\Omega_0\simeq c\Omega_0$) is

$$\left(\frac{dE}{dt}\right)_{\text{Larmor}}=\frac{2}{3}\frac{e^2}{c}\gamma^4\Omega_0^2. \tag{7}$$

Since (1) *must* give exact results in the limit of constant acceleration, our estimate of dE/dt carries a small error of $\sim$ 14% in the normalization constant $9/5\pi$. The reasonable agreement indicates that the method we employed in the evaluation of integrals can reliably be extended to the treatment of more complicated orbits.

We will indeed proceed to calculate the loss rate in a situation where the acceleration is no longer constant. We introduce an orbit of the form (5), but with the replacement $\Omega_0\rightarrow\Omega_0(1+\gamma^2\Omega_1^2\tau^2)^{1/2}$. This is a scenario where the charged particle experiences a spatially nonuniform magnetic field which increases symmetrically before and after the moment $\tau=0$, and equals the minimum value of Ω_0 only at $\tau=0$. As long as the field satisfies $\gamma B\ll B_c=m^2c^3/e\hbar$ everywhere, radiation reaction effects due to particle recoil will remain unimportant. Equation (6), which determines our choice of the time origin t, now reads

$$\phi=\Omega_0 t(1+\gamma^2\Omega_1^2 t^2)^{1/2}. \tag{8}$$

The scenario is interesting because Larmor's radiative loss rate (1) at $t=\tau=0$ is determined by the acceleration at this instance; i.e., it is given by (7) with Ω_0 being the minimum circular frequency. It is intuitively obvious that such a loss rate cannot be correct, since it does not take into account the contributions from the entire coherence length in the neighborhood of the $\tau=0$ position, where more radiation can be expected from the higher acceleration. To demonstrate this in a simple fashion, we will consider the extreme limit $\Omega_1\gg\Omega_0$, when the "beam crossing time" is $\sim 2/\gamma\sqrt{\Omega_0\Omega_1}$, and is governed by the geometric mean of the two frequencies. To $O(1/\gamma^2)$ (2) reads, for this orbit,

$$\frac{d^2E}{d\omega d\Omega}=\frac{e^2\omega^2}{4\pi^2 c}\left[\left|\int_{-\infty}^{\infty}\Omega_0\tau(1+\gamma^2\Omega_1^2\tau^2)^{1/2}e^{i\omega\psi(\tau)}d\tau\right|^2 + \theta^2\left|\int_{-\infty}^{\infty}e^{i\omega\psi(\tau)}d\tau\right|^2\right], \tag{9}$$

where

$$\psi(\tau)=\frac{1}{2}\left[\left(\frac{1}{\gamma^2}+\theta^2\right)\tau+\frac{\Omega_0^2\tau^3}{3}+\frac{\gamma^2\Omega_0^2\Omega_1^2\tau^5}{5}\right].$$

Evaluation of the above integrals is slightly more difficult than before, because there are two complex stationary phases, viz.,

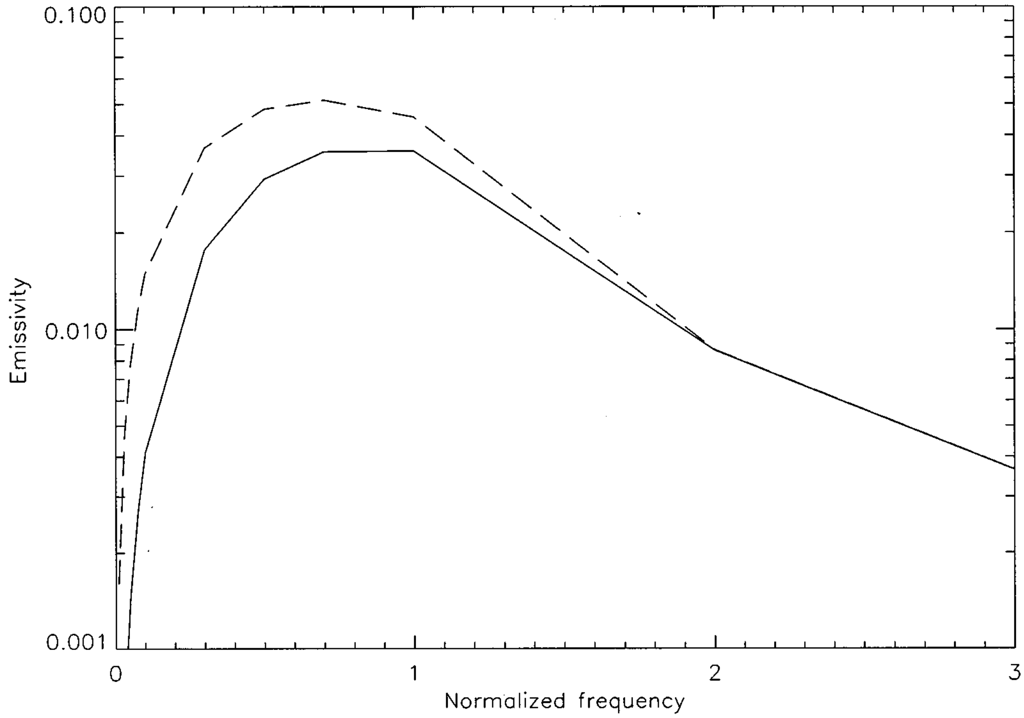

FIG. 1. Variable acceleration synchrotron emissivity for radiation with polarization $(\sin\theta, 0, -\cos\theta)$, as given by the second term of (9) in the limit $\Omega_1 \gg \Omega_0$, plotted as a function of frequency. The radiation propagates in the direction $\theta = 1/\gamma$. The x axis is scaled in units of $\omega/(\gamma^3\sqrt{\Omega_0\Omega_1})$, and the y axis is scaled in units of $e^2\gamma^2/c$. The dashed line represents the stationary phase approximation to the integral, while the solid line represents the exact result as given by the NAG subroutine D01AKF, which is designed to numerically integrate rapidly oscillating integrands with high accuracy.

$$\tau_s = \frac{1}{\sqrt{2\gamma\gamma'\Omega_0\Omega_1}}(m+i), \quad m = \pm 1, \quad \gamma' = \left(\frac{1}{\gamma^2} + \theta^2\right)^{-1/2}$$

and moreover $\ddot{\psi}_r(t_s)$ is now finite. To eliminate the rapid oscillations at $\tau = \tau_s$, the path of steepest descent is chosen to be $y = m(1-\sqrt{2})x$. The large ω limit of (3) is then given by

$$F(\omega) = \sum_{\tau_s} [1 + im(1-\sqrt{2})] \times \left[\frac{\pi|\ddot{\psi}_r|}{\omega(\ddot{\psi}_r^2 + \ddot{\psi}_i^2)}\right]^{1/2} f(\tau_s) e^{-\omega(\psi_i - i\psi_r)},$$

where $\psi_{r,i}$ and their derivatives are all evaluated at $\tau = \tau_s$. This results in the following approximate expression for the emissivity:

$$\frac{d^2E}{d\omega d\Omega} \simeq 0.272 \frac{e^2\omega}{c} (\gamma\Omega_0\Omega_1)^{-1/2} \left(\frac{1}{\gamma^2} + \theta^2\right)^{1/4} \times \left[1 + \frac{\theta^2}{1/\gamma^2 + \theta^2}\right] \times [\cos a\omega + (\sqrt{2}-1)\sin a\omega]^2 e^{-2a\omega}, \quad (10)$$

where $a = (\sqrt{2}/5)(\gamma\Omega_0\Omega_1)^{-1/2}(1/\gamma^2 + \theta^2)^{5/4}$. It is clear that the spectrum extends to frequencies $\sim\gamma^2$ times the reciprocal of the "beam crossing time," as expected from the standard behavior of synchrotron radiation.

In order to assess the errors made, we numerically computed with high accuracy the integrals in (9) and compared them with (10). We find that, as expected, any such errors become negligible at large frequencies where most of the synchrotron radiation power is emitted. At lower frequencies a *constant* systematic offset exists which does not depend on the values of γ, Ω_0, and Ω_1. As illustration, we plot in Fig. 1 the two spectra corresponding to the second integral in (9).

To arrive at a total loss rate we note, as before, that we can integrate (10) over ω and the θ part of the solid angle to produce $dE/d\phi$. $(dE/dt)_{t=0}$ then follows by use of the relation $(d\phi/dt)_{t=0} = \Omega_0$, which is a consequence of (8). The end result is

$$\frac{dE}{dt} = 1.183 \frac{e^2}{c} \gamma^4 \Omega_0^{3/2} \Omega_1^{1/2}.$$

When compared with (7), it is evident that Larmor's formula indeed underestimates the loss rate by a factor $\sim\sqrt{\Omega_1/\Omega_0} \gg 1$. Note that this is a *variable* factor which represents a genuine difference unrelated to the small and constant inaccuracy of our approximation method. It is also conceivable that (7) would overestimate the radiation from orbits which involve less average acceleration than the instantaneous value at $t = 0$.

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